**MATH 5061 Solutions to Problem Set 1**[1]

1. Show that $\mathbb{S}^2$ and $\mathbb{CP}^1$ are diffeomorphic by constructing an explicit diffeomorphism between them.

**Solution:**

Construct the map $f : \mathbb{S}^2 \to \mathbb{CP}^1$ by

$$f(x_1, x_2, x_3) := \begin{cases} [\frac{x_1+ix_2}{1-x_3}, 1], & x_3 \neq 1 \\ [1, \frac{x_1-ix_2}{1+x_3}], & x_3 \neq -1 \end{cases}$$

We need to verify $f$ is well-defined when $x_3 \neq 1, -1$. Indeed, we have (Note $x_1 + ix_2 \neq 0$.)

$$\left[\frac{x_1 + ix_2}{1 - x_3}, 1\right] = \left[1, \frac{1 - x_3}{x_1 + ix_2}\right] = \left[1, \frac{(1 - x_3)(x_1 - ix_2)}{x_1^2 + x_2^2}\right] = \left[1, \frac{x_1 - ix_2}{1 + x_3}\right]$$

which shows $f$ is well-defined.

Now let's show $f$ is a diffeomorphism.

Let $(U_1, \phi_1), (U_2, \phi_2)$ be the two charts on $\mathbb{S}^2$ defined as

$$U_1 = \mathbb{S}^2 \backslash \{(0, 0, 1)\}, \phi_1(x_1, x_2, x_3) = (\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3})$$
$$U_2 = \mathbb{S}^2 \backslash \{(0, 0, -1)\}, \phi_2(x_1, x_2, x_3) = (\frac{x_1}{1 + x_3}, \frac{x_2}{1 + x_3})$$

Let $(V_1, \varphi_1), (V_2, \varphi_2)$ be the two charts on $\mathbb{CP}^1$ defined by

$$V_1 = \mathbb{CP}^1 \backslash \{[1, 0]\}, \varphi_1([z_1, z_2]) = \frac{z_1}{z_2}$$
$$V_2 = \mathbb{CP}^1 \backslash \{[0, 1]\}, \varphi_2([z_1, z_2]) = \frac{z_2}{z_1}$$

So for $p \in U_1$, $f$ has the form under the chart $(U_1, \phi_1)$ and $(V_1, \varphi_1)$ as following

$$\varphi_1 \circ f \circ \phi_1^{-1}(u_1, u_2) = u_1 + iu_2$$

which is a smooth function.

For $p \in U_2$, we have

$$\varphi_2 \circ f \circ \phi_2(u_1, u_2) = u_1 - iu_2$$

which is also smooth.

Hence $f$ is a diffeomorphism.

[1] Last revised on March 5, 2024

2. Prove that the tangent bundle $TM$ is always orientable as a manifold.

**Solution:**

Let $\mathcal{A} = \{(U_i, \phi_i)\}_{i \in I}$ be an atlas of $M^m$. Then we let

$$\tilde{A} := \{(TU_i, \tilde{\phi}_i) : i \in I\} \text{ with } \tilde{\phi}_i(p, v) = (\phi(p), d\phi_p(v)) \in \phi(U_i) \times \mathbb{R}^m$$

The transition maps between $(TU_i, \tilde{\phi}_i), (TU_j, \tilde{\phi}_j)$ is

$$\Phi_{ij}(x, w) = (\phi_j \circ \phi_i^{-1}(x), d(\phi_j \circ \phi_i^{-1})_x(w))$$

Note that $d(\phi_j \circ \phi_i^{-1})_x$ is linear, so the Jacobian matrix is just itself. Hence

$$d\Phi_{ij}(x, w) = \begin{bmatrix} d\left(\phi_j \circ \phi_i^{-1}(x)\right) & 0 \\ 0 & d\left(\phi_j \circ \phi_i^{-1}(x)\right) \end{bmatrix}$$

Hence $\det(d(\Phi_{ij})) = \left[d(\phi_j \circ \phi_i^{-1}(x))\right]^2 > 0$ since $d(\phi_j \circ \phi_i^{-1}(x))$ non-degenerate.

This means all the transition maps are orientation-preserving. Hence $TM$ is orientable.

3. Prove *Jacobi identity*: $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$ for any $X,Y,Z\in\Gamma(TM)$.

**Solution:**

For any $f\in C^\infty(M)$, we directly compute,

$$\begin{aligned}[X,[Y,Z]]f &= X([Y,Z]f)-[Y,Z](Xf)\\ &= X(YZf-ZYf)-YZXf+ZYXf\\ &= XYZf-YZXf+XZYf-ZYXf\end{aligned}$$

Similarly

$$\begin{aligned}[Y,[Z,X]]f &= YZXf-ZXYf+YXZf-XZYf\\ [Z,[X,Y]]f &= ZXYf-XYZf+ZYXf-YXZf\end{aligned}$$

Adding them up

$$\begin{aligned}&[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]f\\ =\ &(XYZ+YZX+ZXY)f-(YZX+ZXY+XYZ)f\\ &(XZY+YXZ+ZYX)f-(ZYX+XZY+YXZ)f\\ =\ &0\end{aligned}$$

Hence

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$$

4. Let $\alpha$ be a $(0,q)$-tensor on $M$, $X, Y_1, \cdots, Y_q \in \Gamma(TM)$ be vector fields. Show that

$$(\mathcal{L}_X\alpha)(Y_1,\cdots,Y_q) = X(\alpha(Y_1,\cdots,Y_q)) - \sum_{i=1}^{q}\alpha(Y_1,\cdots,Y_{i-1},[X,Y_i],Y_{i+1},\cdots,Y_q).$$

**Solution:**

By definition of pull-back, we have

$$(\phi_t^*\alpha)(Y_1,\cdots,Y_q)(x) = \alpha_{\phi_t(x)}(\phi_{t*}Y_1,\cdots,\phi_{t*}Y_q)$$

with $x \in M$ where $\phi_t$ is the flow generated by $X$.

So

$$\begin{aligned}
&(\mathcal{L}_X\alpha)(Y_1,\cdots,Y_q)(x)\\
=&\lim_{t\to 0}\frac{1}{t}\left(\left(\phi_t^*\alpha\right)(Y_1,\cdots,Y_q)(x) - \alpha_x(Y_1,\cdots,Y_q)\right)\\
=&\lim_{t\to 0}\frac{1}{t}\left(\alpha_{\phi_t(x)}(\phi_{t*}Y_1,\cdots,\phi_{t*}Y_q) - \alpha_x(\phi_{t*}Y_1,\cdots,\phi_{t*}Y_q)\right)\\
&+\sum_{i=1}^{q}\lim_{t\to 0}\frac{1}{t}\left[\alpha_x(\phi_{t*}Y_1,\cdots,\phi_{t*}Y_{i-1},\phi_{t*}Y_i,Y_{i+1},\cdots,Y_q)\right.\\
&-\left.\alpha_x(\phi_{t*}Y_1,\cdots,\phi_{t*}Y_{i-1},Y_i,Y_{i+1},\cdots,Y_q)\right]\\
=&X(\alpha(Y_1,\cdots,Y_q))(x) + \sum_{i=1}^{q}\alpha_x(Y_1,\cdots,Y_{i-1},\mathcal{L}_XY_i,Y_{i+1},\cdots,Y_q)\\
=&X(\alpha(Y_1,\cdots,Y_q))(x) - \sum_{i=1}^{q}\alpha_x(Y_1,\cdots,Y_{i-1},[X,Y_i],Y_{i+1},\cdots,Y_q)
\end{aligned}$$

Since the above identity holds for all $x \in M$, we have

$$(\mathcal{L}_X\alpha)(Y_1,\cdots,Y_q) = X(\alpha(Y_1,\cdots,Y_q)) - \sum_{i=1}^{q}\alpha(Y_1,\cdots,Y_{i-1},[X,Y_i],Y_{i+1},\cdots,Y_q)$$